

IDEAL DECOMPOSITIONS OF KILLING AND HOLOMORPHIC VECTOR FIELDS

WALTER C. LYNGE

1. Introduction

Let (M, g) be a compact, connected, oriented Riemannian manifold. We show in § 3 that the Lie algebra of Killing vector fields on M can be decomposed into a direct sum of ideals according to the reducibility of the linear holonomy group of M . A decomposition of this type is already known for the case of a simply connected, complete Riemannian manifold. If in addition M is assumed to be Kählerian, we show in § 4 that the Lie algebra of holomorphic vector fields on M can also be decomposed in this way. Our proofs make use of part of a theory of Chern in the form described briefly in § 2.

2. Preliminaries

Let $O(M)$ and $\Gamma_{O(M)}$ denote respectively the oriented orthonormal frame bundle over M with structure group $SO(n)$ and the Riemannian connection in $O(M)$. We assume $\Gamma_{O(M)}$ to be reducible to a connection Γ_P in a subbundle P of $O(M)$ with structure group $G \subset SO(n)$ and projection π . The case which will interest us most in the sequel is where P is the holonomy bundle through some point of $O(M)$.

Let $\{e_1, \dots, e_n\}$ and $\{A_1, \dots, A_n\}$ be respectively the canonical basis of R^n and a basis for the Lie algebra of G . Let $\theta = \sum_{i=1}^n \theta^i e_i$, $\omega = \sum_{\lambda=1}^m \omega^\lambda A_\lambda$, and $\Omega = \sum_{\lambda=1}^m \Omega^\lambda A_\lambda$ be respectively the canonical form of P , the connection form of Γ_P , and the curvature form of Γ_P . We have the formula $\Omega^\lambda = \frac{1}{2} \sum_{i,j=1}^n r_{ij}^\lambda \theta^i \wedge \theta^j$ with $r_{ij}^\lambda = -r_{ji}^\lambda$. Let $(A_\lambda)_{ij} = a_{ij}^\lambda$, $i, j = 1, \dots, n$. Then there exist functions $s^{\lambda\mu}$ on P such that $r_{ki}^\lambda = \sum_{\mu} s^{\lambda\mu} a_{ik}^\mu$ with $s^{\lambda\mu} = s^{\mu\lambda}$. Let A_λ^* denote the fundamental vector field on P corresponding to A_λ , $\lambda = 1, \dots, m$. Let X_i , $i = 1, \dots, n$, be vector fields on P such that $\theta^i(X_j) = \delta_j^i$, $\omega^\lambda(X_j) = 0$. Define differential operators P^* and S^* on the space of smooth functions on P by $P^* = \sum_k X_k^2$ and $S^* = \sum_{\mu, \lambda} s^{\lambda\mu} A_\mu^* A_\lambda^*$. These differential operators commute with right translation by

Communicated by Y. Matsushima, December 3, 1971.

elements of G . Thus, if ρ is a representation of G in a vector space V , and f is a V -valued equivariant function on P , then P^*f and S^*f are also equivariant, where P^*f and S^*f are defined componentwise for V -valued functions after selecting a basis for V .

Let j denote the natural representation of G in R^n . Let $\mathfrak{F} = \{\text{equivariant smooth functions on } P \text{ of type } (j, R^n)\}$, $\mathfrak{D} = \{\text{smooth 1-forms on } M\}$, and $\mathfrak{X} = \{\text{smooth vector fields on } M\}$. Given η in \mathfrak{D} , we have $\pi^*\eta = \sum_{i=1}^n f_i \theta^i$. To η we associate $f = \sum_{i=1}^n f_i e^i$ in \mathfrak{F} . This defines a 1:1 correspondence between \mathfrak{D} and \mathfrak{F} . Moreover, the Riemannian metric gives a 1:1 correspondence between \mathfrak{D} and \mathfrak{X} . Hence we have also a 1:1 correspondence between \mathfrak{X} and \mathfrak{F} .

Let Δ and S be respectively the Laplacian on M and the Ricci tensor interpreted as an endomorphism of the cotangent space at each point. By the 1:1 correspondence between \mathfrak{D} and \mathfrak{F} , we may regard the maps $\Delta, S: \mathfrak{D} \rightarrow \mathfrak{D}$ as maps $\Delta, S: \mathfrak{F} \rightarrow \mathfrak{F}$. In terms of the operators P^* and S^* , we then have the expressions $\Delta = -P^* - S^*$ and $S = S^*$.

Suppose c is a linear transformation of R^n which commutes with the action of G . Let η be an element of \mathfrak{F} . Define $(c \cdot \eta)(u) = c(\eta(u))$ for each u in P . Then $c \cdot \eta$ is in \mathfrak{F} , and we have a map $\mathfrak{F} \rightarrow \mathfrak{F}$ also denoted by c . By a theorem of Chern the diagram

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{c} & \mathfrak{F} \\ T \downarrow & & \downarrow T \\ \mathfrak{F} & \xrightarrow{c} & \mathfrak{F} \end{array}$$

commutes when T is P^* or S^* .

3. Decomposition of Killing vector fields

Our decomposition theorem for Killing vector fields is:

Theorem 1. *Let M be a compact, connected, oriented Riemannian manifold with oriented orthonormal frame bundle $O(M)$. Let P be any holonomy bundle of $O(M)$ with structure group Φ . Suppose $R^n = V_1 \oplus \dots \oplus V_r$, where the V_i are mutually orthogonal subspaces of R^n with respect to the usual inner product and each V_i is invariant under Φ . Let \mathfrak{G} be the Lie algebra of Killing vector fields on M . Then $\mathfrak{G} = \mathfrak{G}_1 \oplus \dots \oplus \mathfrak{G}_r$, where $\mathfrak{G}_i = \{X | X \in \mathfrak{G} \text{ and the equivariant } R^n\text{-valued function on } P \text{ corresponding to } X \text{ is } V_i\text{-valued}\}$. Moreover, each \mathfrak{G}_i is an ideal in \mathfrak{G} .*

Proof. We can find an element g in $SO(n)$ such that $g^{-1}V_i = R^{p_i}$, $i = 1, \dots, r$, where R^{p_i} is the subspace of R^n spanned by $\{e_1, \dots, e_{p_i}\}$, R^{p_2} is the subspace spanned by $\{e_{p_1+1}, \dots, e_{p_1+p_2}\}$, etc. Let P' be the holonomy bundle

$R_g P$ with structure group $g^{-1}\Phi g = \Phi'$. Then Φ' leaves each R^{p_i} invariant, and $R^n = R^{p_1} \oplus \dots \oplus R^{p_r}$. Let $\mathfrak{G}'_i = \{X \mid X \in \mathfrak{G} \text{ and the equivariant } R^n\text{-valued function on } P' \text{ corresponding to } X \text{ is } R^{p_i}\text{-valued}\}$. It is easy to compute that $\mathfrak{G}_i = \mathfrak{G}'_i$. Thus we may consider V_i to be R^{p_i} , $i = 1, \dots, r$. Furthermore, we restrict ourselves to the case $r = 2$ for simplicity and write $R^{p_1} = R^p$, $R^{p_2} = R^{n-p}$.

Let X be a Killing vector field on M with corresponding 1-form α . Let f be the R^n -valued function on P corresponding to X . Let ρ_1, ρ_2 be the projection maps $R^n \rightarrow R^p$ and $R^n \rightarrow R^{n-p}$ respectively. Then $f = f_1 + f_2$, where $f_i = \rho_i f$, $i = 1, 2$. Let X_1, X_2 be the vector fields on M corresponding to f_1, f_2 respectively. Then $X = X_1 + X_2$. If we show X_1 and X_2 are Killing, it follows easily that $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$. Let α_1, α_2 be the 1-forms on M corresponding to X_1, X_2 respectively. Applying the Chern theorem, we see that $\Delta\alpha_i = 2S\alpha_i$, $i = 1, 2$. To show X_1 and X_2 are Killing, it suffices now to prove that $\delta\alpha_i = 0$, $i = 1, 2$. In order to do this, we first make a felicitous choice of moving frame:

Let v be an element of P with $\pi(v) = x$. Write $v = (Y_{1x}, \dots, Y_{nx})$, where $Y_{ix} \in T_x M$, $i = 1, \dots, n$. Let $\{x^1, \dots, x^n\}$ be the normal coordinate system on a neighborhood U of x determined by the frame v . Then $\partial/\partial x^i|_x = Y_{ix}$, $i = 1, \dots, n$. Translate v parallelly along geodesics emanating from x to obtain a moving orthonormal frame $\sigma = [Y_1, \dots, Y_n]$ over U . By the definition of P , σ is a section of P over U . It is called the adapted frame on a neighborhood U of x determined by v . Define two distributions V and W on U by $V_y =$ subspace of $T_y M$ spanned by $\{Y_{1y}, \dots, Y_{py}\}$ and $W_y =$ subspace spanned by $\{Y_{p+1y}, \dots, Y_{ny}\}$, $y \in U$. We have that $T_y M = V_y \oplus W_y$ and $V_y \perp W_y$ for each y in U .

Lemma 1. V_y and W_y are invariant under $\Phi(y)$, where $\Phi(y)$ is the linear holonomy group of $\Gamma_{O(M)}$ with reference point y regarded as a linear group acting on $T_y M$.

Proof. We recall that the action of $\Phi(y)$ on $T_y M$ may be described as follows: Let $w = (Y_{1y}, \dots, Y_{ny})$, and suppose τ is an element of $\Phi(y)$. Regarding the linear frames w and $\tau(w)$ as maps from R^n to $T_y M$ in the usual way, we associate to τ the linear transformation $\tau(w) \circ w^{-1}$ of $T_y M$. We need to show that $\tau(w) \circ w^{-1}$ leaves V_y and W_y invariant. Let $\Phi(w)$ denote the holonomy group of $\Gamma_{O(M)}$ with reference point w . Since $w \in P$, $\Phi(w) = \Phi$. Thus there is an element g in Φ such that $\tau(w)$ is the linear frame $w \circ g: R^n \rightarrow T_y M$. But g leaves R^p and R^{n-p} invariant. Moreover $w(R^p) = V_y$ and $w(R^{n-p}) = W_y$. Hence the result.

Next we need to establish some properties of the Christoffel symbols Γ^i_{jk} with respect to the frame σ . We do this in the next four lemmas, omitting most of the details.

Lemma 2. $\Gamma^k_{ij} = 0$ if $k > p$ and $i, j \leq p$, and $\Gamma^k_{ij} = 0$ if $k \leq p$ and $i, j > p$, everywhere on U .

Proof. Using Lemma 1 we can show that if X and Z are vector fields on

U belonging to the distribution V (respectively W), then $\nabla_Z X$ and $\nabla_X Z$ also belong to V (respectively W). The result then follows at once.

Lemma 3. $\Gamma_{mi}^j = -\Gamma_{mj}^i \forall i, m, j$ everywhere on U .

Proof. This follows from the fact that $\nabla g = 0$ and the fact that the moving frame $[Y_1, \dots, Y_n]$ is orthonormal.

Lemma 4. $\Gamma_{ij}^k = \Gamma_{ji}^k$ at $x \forall i, j, k$.

Proof. Let $Y = \sum_{i=1}^n y^i \partial/\partial x^i$ be the vector field on U determined by parallelly translating a vector Y_x at x along geodesics emanating from x , where again $\{x^1, \dots, x^n\}$ are normal coordinates on U determined by v . Using the differential equations for parallel translation, we can show that $\partial y^k/\partial x^l = 0$ at $x \forall k, l$. We then apply this fact to show that $[Y_i, Y_j](x) = 0 \forall i, j$. Since the torsion is zero, we then have the result.

Lemma 5. $\Gamma_{im}^k = 0$ at x unless k, i, m are all $\leq p$ or all $> p$.

Proof. Immediate from Lemmas 2, 3, 4.

We now return to the proof that $\delta\alpha_i = 0, i = 1, 2$. Pick an arbitrary point x in M . Let v be an element of P such that $\pi(v) = x$. Let $[Y_1, \dots, Y_n] = \sigma$ be the adapted frame on a neighborhood U of x determined by v . Let $[\beta_1, \dots, \beta_n]$ be the moving coframe dual to σ . It is not difficult to check that we have representations of the form $\alpha_1 = \sum_{i=1}^p \eta_i \beta^i$ and $\alpha_2 = \sum_{i=p+1}^n \eta_i \beta^i$. Then $X_1 = \sum_{i=1}^p \eta^i Y_i$ and $X_2 = \sum_{i=p+1}^n \eta^i Y_i$, where $\eta^i = \eta_i, i = 1, \dots, n$, because σ is an orthonormal frame. Since X is Killing, we have $\eta_{j,i} + \eta_{i,j} = 0 \forall i, j$. Setting $i = j$, we have $\eta_{j,j} = 0$. But $\eta_{j,j} = Y_j(\eta_j) - \sum_i \eta_i \Gamma_{jj}^i$. Hence $Y_j(\eta_j) = \sum_i \eta_i \Gamma_{jj}^i$. By Lemma 5 we have

$$(1) \quad \text{if } j \leq p, \quad \text{then } Y_j(\eta_j)(x) = \sum_{i \leq p} \eta_i(x) \Gamma_{jj}^i(x),$$

$$(2) \quad \text{if } j > p, \quad \text{then } Y_j(\eta_j)(x) = \sum_{i > p} \eta_i(x) \Gamma_{jj}^i(x).$$

We recall that $\eta_{j,j}^i = Y_j(\eta_j^i) + \sum_k \Gamma_{jk}^i \eta^k$. From this, using (1), (2), Lemmas 3 and 5, and the fact that $\eta^i = \eta_i$ it can be computed that $\eta_{j,j}^i(x) = 0$ for each $j = 1, \dots, n$. But $(\delta\alpha_1)(x) = -\sum_{j=1}^p \eta_{j,j}^i(x)$ and $(\delta\alpha_2)(x) = -\sum_{j=p+1}^n \eta_{j,j}^i(x)$. Hence $(\delta\alpha_1)(x) = (\delta\alpha_2)(x) = 0$ for an arbitrary point x in M . Thus X_1 and X_2 are Killing.

It remains only to show that \mathfrak{G}_1 and \mathfrak{G}_2 are ideals in \mathfrak{G} . First we show $[\mathfrak{G}_1, \mathfrak{G}_1] \subset \mathfrak{G}_1$. Let X, Z be arbitrary elements of \mathfrak{G}_1 , and v be an arbitrary point in P with $\pi(v) = x$. Let $\sigma = [Y_1, \dots, Y_n]$ be the adapted frame on a neighborhood U of x determined by v . We see easily that X and Z are expressed as

$X = \sum_{i=1}^n \eta^i Y_i, Z = \sum_{i=1}^n \xi^i Y_i$, where $\eta^i = \xi^i \equiv 0$ on U for $i > p$. Now $[X, Z] = \sum_{m=1}^n [X, Z]^m Y_m$ where

$$(3) \quad [X, Z]^m = \sum_{i,j} (\xi^i Y_i(\eta^j) + \xi^i \eta^j \Gamma_{ij}^m - \eta^i Y_i(\xi^j) - \eta^i \xi^j \Gamma_{ji}^m).$$

By Lemma 5 we obtain that $[X, Z]^m(x) = 0$ for $m > p$. Thus, if $\alpha_i, i = 1, \dots, n$, are the components of the 1-form corresponding to $[X, Z]$, then $\alpha_i(x) = 0$ for $i > p$. From this it follows that if $f = \sum_{i=1}^n f^i e_i$ is the R^n -valued function on P corresponding to $[X, Z]$, then $f^i(v) = 0$ for $i > p$. But v was an arbitrary point of P . Hence $[X, Z] \in \mathfrak{G}_1$. Similarly we prove that $[\mathfrak{G}_2, \mathfrak{G}_2] \in \mathfrak{G}_2$.

To complete the proof of Theorem 1, we need only to show $[\mathfrak{G}_1, \mathfrak{G}_2] = 0$. Let X, Z be arbitrary elements of $\mathfrak{G}_1, \mathfrak{G}_2$ respectively. Then $X = \sum_{i=1}^n \xi^i Y_i$ and $Z = \sum_{i=1}^n \eta^i Y_i$ with $\xi^i \equiv 0$ on U for $i > p$ and $\eta^i \equiv 0$ on U for $i \leq p$, where $\sigma = [Y_1, \dots, Y_n]$ is an adapted frame on a neighborhood U of an arbitrary point x of M determined by some v in P such that $\pi(v) = x$. Again we have $[X, Z] = \sum_{m=1}^n [X, Z]^m Y_m$ where $[X, Z]^m$ is given by (3). We will show that $[X, Z]^m(x) = 0 \forall m$.

Case 1: $m > p$. Then $[X, Z]^m(x) = \sum_{i=1}^n \xi^i(x)(Y_i(\eta^m))(x)$ by Lemma 4 and the fact that $\xi^i \equiv 0$ on U for $i > p$. Since Z is Killing, we have $\eta_{m,i} + \eta_{i,m} = 0$. Since σ is orthonormal, $\eta^i = \eta_i$. Thus we obtain

$$Y_i(\eta^m) - \sum_k \eta_k \Gamma_{mi}^k + Y_m(\eta^i) - \sum_k \eta_k \Gamma_{im}^k = 0.$$

For $m > p$ and $i \leq p$, it follows by Lemma 5 that $Y_i(\eta^m) = -Y_m(\eta^i)$ at x . Thus $[X, Z]^m(x) = -\sum_{i \leq p} \xi^i(x)(Y_m(\eta^i))(x) = 0$.

Case 2: $m \leq p$. The proof is similar.

4. Decomposition of holomorphic vector fields

We now assume our compact, connected, oriented Riemannian manifold M is Kählerian. The complex dimension is $n/2$, where n is now even. We denote the complex structure by J . Let \mathfrak{A} be the real Lie algebra of infinitesimal automorphisms of M . \mathfrak{A} is made into a complex Lie algebra using the almost complex structure J . Let \mathfrak{S} be the complex Lie algebra of holomorphic vector fields on M . As complex Lie algebras, $\mathfrak{S} = \mathfrak{A}$. Thus we will deal with \mathfrak{A} from now on.

The infinitesimal automorphisms of M are characterized by the property that

their corresponding 1-forms satisfy $\Delta\alpha = 2S\alpha$. Therefore, if $F_{O(M)}$ is reducible to a connection F_P in a subbundle P of $O(M)$ with structure group $G \subset SO(n)$, and if $R^n = V_1 \oplus \dots \oplus V_r$ where each V_i is invariant under the action of G , then by the theorem of Chern we have a corresponding decomposition of the real Lie algebra \mathfrak{A} into a direct sum of vector subspaces: $\mathfrak{A} = \mathfrak{A}_1 \oplus \dots \oplus \mathfrak{A}_r$, where $\mathfrak{A}_i = \{X \mid X \in \mathfrak{A} \text{ and the equivariant } R^n\text{-valued function on } P \text{ corresponding to } X \text{ is } V_i\text{-valued}\}$.

Now we specialize P and G . Let x be an arbitrary point of M . Let $T_xM = V_{1x} \oplus \dots \oplus V_{sx}$ be a direct sum decomposition of T_xM such that the V_{ix} are mutually orthogonal subspaces invariant under $\Phi(x)$ and J_x . There is an element u in $O(M)$, $\pi(u) = x$, which is of the form

$$u = (u_1, \dots, u_{r_1}, Ju_1, \dots, Ju_{r_1}, u_{r_1+1}, \dots, u_{r_1+r_2}, Ju_{r_1+1}, \dots, Ju_{r_1+r_2}, \dots)$$

where the first $2r_1$ vectors in the frame u span V_{1x} , the next $2r_2$ vectors span V_{2x} , etc.

Theorem 2. *Let $P = P(u)$ be the holonomy bundle of $O(M)$ through u with structure group $G = \Phi(u) = \Phi$. Let $R^n = R^{2r_1} \oplus \dots \oplus R^{2r_s}$. Then each R^{2r_i} , $i = 1, \dots, s$, is invariant under Φ . Moreover, if $\mathfrak{A} = \mathfrak{A}_1 \oplus \dots \oplus \mathfrak{A}_s$ is the decomposition of \mathfrak{A} corresponding to this decomposition of R^n , then the \mathfrak{A}_i are complex subspaces of \mathfrak{A} and in fact ideals.*

Proof. For simplicity we assume $s = 2$ and write $R^{2r_1} = R^p$, $R^{2r_2} = R^{n-p}$, $V_{1x} = V_x$, and $V_{2x} = W_x$. To show R^p is invariant under Φ , let g be an arbitrary element of Φ . Then there is an element τ of $\Phi(x)$ such that $\tau(u)$ is the frame- $u \circ g: R^n \rightarrow T_xM$. Since V_x is invariant under $\Phi(x)$, we have $\tau(u) \circ u^{-1}V_x = u \circ g \circ u^{-1}V_x = V_x$. But $u^{-1}V_x = R^p$. Therefore $u \circ gR^p = V_x$ and $gR^p = u^{-1}V_x = R^p$. Similarly R^{n-p} is invariant under Φ .

Now let v be an arbitrary element of P with $\pi(v) = y$. By the definition of $P = P(u)$ and the fact that $\nabla J = 0$ we see that v is of the form

$$v = (v_1, \dots, v_{r_1}, Jv_1, \dots, Jv_{r_1}, v_{r_1+1}, \dots, v_{r_1+r_2}, Jv_{r_1+1}, \dots, Jv_{r_1+r_2})$$

where $2r_1 = p$ and $2r_2 = n - p$. Thus, if V_y is the subspace of T_yM spanned by the first p vectors of this frame and W_y is the subspace spanned by the last $n - p$ vectors, then we have $J_yV_y = V_y$ and $J_yW_y = W_y$. It follows readily that $J\mathfrak{A}_i \subset \mathfrak{A}_i$, $i = 1, 2$. Thus \mathfrak{A}_1 and \mathfrak{A}_2 are complex subspaces of \mathfrak{A} .

It remains to show that \mathfrak{A}_1 and \mathfrak{A}_2 are ideals in \mathfrak{A} . The proof that $[\mathfrak{A}_i, \mathfrak{A}_i] \subset \mathfrak{A}_i$, $i = 1, 2$, is exactly the same as the proof of the analogous fact in Theorem 1. We therefore have only to show $[\mathfrak{A}_1, \mathfrak{A}_2] = 0$. Let X, Z be arbitrary elements of $\mathfrak{A}_1, \mathfrak{A}_2$ respectively. Let y be an arbitrary point of M and let $\sigma = [Y_1, \dots, Y_n]$ be an adapted frame on a neighborhood U of y determined by an element $v = (Y_{1y}, \dots, Y_{ny})$ in P . Let the components of X, Z , and $[X, Z]$ be ξ^i, η^i and $[X, Z]^m$ respectively in the frame σ . Then $\xi^i \equiv 0$ for $i > p$ and $\eta^i \equiv 0$ for $i \leq p$. From Lemma 4 we obtain

$$[X, Z]^m(y) = \sum_{i=1}^n \xi^i(y)(Y_i(\eta^m))(y) - \eta^i(y)(Y_i(\xi^m))(y).$$

Thus $[X, Z]^m(y) = -\sum_{i>p} \eta^i(y)(Y_i(\xi^m))(y)$ for $m \leq p$, and $[X, Z]^m(y) = \sum_{i \leq p} \xi^i(y)(Y_i(\eta^m))(y)$ for $m > p$. In order to show $[X, Z]^m(y) = 0 \forall m$, it suffices to show

$$\begin{aligned} (Y_i(\xi^m))(y) &= 0 \quad \text{for } i > p \text{ and } m \leq p, \text{ and} \\ (Y_i(\eta^m))(y) &= 0 \quad \text{for } i \leq p \text{ and } m > p. \end{aligned}$$

Lemma 6. $\Phi^0(y)$, the restricted linear holonomy group at y (which is the identity component of $\Phi(y)$) with respect to $\Gamma_{O(M)}$, is decomposed into the direct product of two normal subgroups $\Phi_1^0(y)$ and $\Phi_2^0(y)$ such that $\Phi_1^0(y)$ is trivial on W_y and $\Phi_2^0(y)$ is trivial on V_y .

Proof. This follows from the proof of Proposition 5.3, p. 183 in [2]. By Lemma 1 we have that $\Phi_1^0(y)V_y \subset V_y$ and $\Phi_2^0(y)W_y \subset W_y$. From this and Lemma 6 it follows that the holonomy algebra $\phi(y)$ at y splits into a direct sum of subalgebras $\phi_1(y)$ and $\phi_2(y)$, where, as linear endomorphisms of T_yM with respect to the basis $\{Y_{1y}, \dots, Y_{ny}\}$, elements of $\phi_1(y)$ (respectively $\phi_2(y)$) are represented by matrices of the form

$$\begin{pmatrix} a_{p \times p} & 0 \\ 0 & 0 \end{pmatrix} \left(\text{respectively } \begin{pmatrix} 0 & 0 \\ 0 & b_{(n-p) \times (n-p)} \end{pmatrix} \right).$$

Now let η be the 1-form corresponding to the infinitesimal automorphism Z . By an easy computation we find that with respect to the moving frame σ , $(\nabla\eta)_{li} = Y_i(\eta^l) + \sum_j \eta^j \Gamma_{ij}^l$. In particular, $(\nabla\eta)_{li}(y) = (Y_i(\eta^l))(y)$ if $l > p$ and $i \leq p$. But an element of $\phi(y) + J_y\phi(y)$ is represented with respect to the basis $\{Y_{1y}, \dots, Y_{ny}\}$ of T_yM by a matrix of the form

$$\begin{pmatrix} A_{p \times p} & O_{p \times (n-p)} \\ O_{(n-p) \times p} & B_{(n-p) \times (n-p)} \end{pmatrix}.$$

This follows from the above representations of $\phi_i(y)$, $i = 1, 2$, and from the fact that $J_yV_y = V_y$ and $J_yW_y = W_y$. By a theorem of Lichnerowicz [3, p. 151], $(\nabla\eta)_y \in \phi(y) + J_y\phi(y)$. This implies that $(\nabla\eta)_{li}(y) = 0$ if $l > p$ and $i \leq p$. Thus we must have $(Y_i(\eta^m))(y) = 0$ for $i \leq p$ and $m > p$. Similarly, we can prove that $(Y_i(\xi^m))(y) = 0$ for $i > p$ and $m \leq p$. This completes the proof of Theorem 2.

Finally, if we assume the Kähler manifold M is nondegenerate, we can obtain a decomposition of \mathfrak{A} starting, as in the Killing case, from any holonomy bundle :

Theorem 3. *Let M be nondegenerate. Suppose P is any holonomy bundle of $O(M)$, and $G = \Phi$ is its structure group. Suppose $R^n = V_1 \oplus \cdots \oplus V_r$ is a decomposition of R^n into a direct sum of subspaces mutually orthogonal with respect to the usual inner product and invariant under Φ . Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ be the corresponding decomposition of \mathfrak{A} . Then the \mathfrak{A}_i are complex subspaces of \mathfrak{A} and in fact ideals.*

Proof. As in Theorem 1, it suffices to consider $V_1 \oplus \cdots \oplus V_r$ to be $R^{p_1} \oplus \cdots \oplus R^{p_r}$. For simplicity, we consider only the case $r=2$ and write $R^n = R^p \oplus R^{n-p}$. Let $u = (Y_{1x}, \cdots, Y_{nx})$ be an element of P with $\pi(u) = x$. Then by Lemma 1, V_x and W_x , the subspaces of $T_x M$ spanned by $\{Y_{1x}, \cdots, Y_{px}\}$ and $\{Y_{p+1x}, \cdots, Y_{nx}\}$ respectively, are invariant under $\Phi(x)$. Since M is nondegenerate, we have $J_x \in \Phi(x)$ [3, p. 173]. Then, in particular, $JV_x \subset V_x$ and $JW_x \subset W_x$. The rest of the proof proceeds as the proof of Theorem 2.

The results of this paper constitute part of the author's Ph. D. dissertation written at the University of Notre Dame. The author would like to express his gratitude to Professor Yozo Matsushima for much encouragement and many helpful suggestions.

Bibliography

- [1] S. S. Chern, *On a generalization of Kähler geometry*, Algebraic geometry and Topology, a Symposium in Honor of S. Lefschetz, Princeton University Press, Princeton, 1957, 103-121.
- [2] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963.
- [3] ———, *Foundations of differential geometry*, Vol. II, Interscience, New York, 1969.
- [4] A. Lichnerowicz, *Géométrie des groupes de transformations*, Dunod, Paris, 1958.
- [5] A. Weil, *Un théorème fondamental de Chern en géométrie Riemannienne*, Séminaire Bourbaki, 14ième année, 1961-62, Exp. 239.

EDGECLIFF COLLEGE